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Extraconnectivity of s-geodetic digraphs and graphs

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Abstract

The η-extraconnectivity κ_η of a simple connected (di)graph G is a kind of conditional connectivity which is defined as the minimum cardinality of a set of vertices, whose deletion disconnects G, in such a way that all remaining (strongly) connected components have cardinality greater than η. The usual connectivity and superconnectivity of G correspond to κ_0 and κ_1, respectively. First, this paper gives sufficient conditions, relating the diameter D, the parameter s and the minimum degree δ of a s-geodetic digraph, to assure maximum η-extraconnectivity. To be more precise, it is proved that if $D \leqslant 2s-1-2\lfloor \log_\delta(\eta(\delta-1)+1) \rfloor$, being $\eta \geqslant 2$ and $s \geqslant \eta+1$, then the value of κ_η is $(\eta+1)(\delta-1)$, which is optimal. Finally, in the undirected case, it is proved, for instance, that if

$$D \leqslant g-6-4\left\lfloor \log_{(\delta-1)}\left(\frac{\eta(\delta-2)+2}{\delta}\right)\right\rfloor,$$

being $g \geqslant \eta+5$ the girth of graph, $\eta \geqslant \delta+1$, and $\delta \geqslant 5$, then the value of κ_η is $(\eta+1)\delta-2\eta$, which is optimal. So, knowing the sufficient conditions relating the diameter D, the girth g and the minimum degree δ of a graph, to assure maximum η-extraconnectivity are improved for $\eta \geqslant 2(\delta+1)$. The corresponding edge version of these results, to assure maximum edge η-extraconnectivity λ_η, are also discussed.

1. Introduction

One of the most important properties to be taken into account when designing an interconnection network is its fault-tolerance; that is, the ability of the system to work even if some nodes and/or links fail. See the survey of Bermond, Homobono and Peyrat [3]. This paper is devoted to the study of (di)graph models for optimally connected networks with respect to the following fault-tolerance property: when some

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nodes or links fail, all surviving components of the network have to connect a given minimum number of nodes. This problem corresponds to the study of a kind of conditional (di)graph connectivity introduced by Harary in [10].

The standard graph theoretic terms not defined in this paper can be found in the book of Chartrand and Lesniak [6]. In particular, the following concepts and notations are used in this paper. Thus, G stands for a (finite) simple digraph, that is, without loops or multiple edges, with set of vertices $V=V(G)$ and set of (directed) edges $E=E(G)$. For any pair of vertices $x,y\in V$, a path from x to y is called an $x\to y$ path. A digraph G is said to be strongly connected when for any pair of vertices $x,y\in V$ there always exists an $x\to y$ path. We always assume that our digraphs are (strongly) connected and that they are different from a complete digraph and so, the diameter is $D>1$. The distance from x to a subset of vertices $F\subset V$, denoted by $d(x,F)$, is the minimum over all the distances $d(x,y)$, $y\in F$. The distance from F to vertex x is defined analogously. Given a subset of edges $A\subset E$ we define $d(x,A)=\min_{(u,v)\in A} d(x,u)$, and $d(A,x)=\min_{(u,v)\in A} d(v,x)$.

Given a proper (nonempty) subset of vertices $F\subset V$, let $\Gamma^+(F)=\bigcup_{x\in F}\Gamma^+(x)$ and $\Gamma^-(F)=\bigcup_{x\in F}\Gamma^-(x)$. The *positive* and *negative boundaries* of F are $\partial^+F=\Gamma^+(F)\backslash F$ and $\partial^-F=\Gamma^-(F)\backslash F$. The corresponding concepts for edges are the *positive* and *negative edge-boundaries*, $\omega^+F=\{(x,y)\in E\colon x\in F \text{ and } y\in V\backslash F\}$ and $\omega^-F=\{(x,y)\in E\colon x\in V\backslash F \text{ and } y\in F\}$.

Clearly, if $F\cup\partial^+F\neq V$ $[F\cup\partial^-F\neq V]$ then ∂^+F $[\partial^-F]$ is a cutset of G. Similarly, if F is a proper (nonempty) subset of V, then ω^+F $[\omega^-F]$ is an edge-cutset. Obviously, for any $x\in V$, $\partial^+\{x\}$ $[\partial^-\{x\}]$ is a trivial cutset, and $\omega^+\{x\}$ $[\omega^-\{x\}]$ is a trivial edge-cutset as well. Next we introduce new concepts, which are a generalization of these sets.

Given an integer $\eta\geqslant 0$ and a subset of vertices $F\subset V$, let us say that ∂^+F $[\partial^-F]$ is *η-trivial* if there exists a subset $F'\subset F$, $F'\neq\emptyset$, with cardinality at most η, such that ∂^+F $[\partial^-F]$ contains the set ∂^+F' or ∂^-F'. Analogously, the set of edges ω^+F $[\omega^-F]$ is said to be *η-trivial* if ω^+F $[\omega^-F]$ contains a set ω^+F' or ω^-F', for some $F'\subset F$ with $|F'|\leqslant\eta$. Notice that for $\eta=0$ each subset of vertices is 0-nontrivial.

Hence, by using these concepts, we define the new parameters

$$\kappa_\eta=\min\{|\partial^+F|:\partial^+F \ \eta\text{-nontrivial},\ F\cup\partial^+F\neq V\},$$
$$\lambda_\eta=\min\{|\omega^+F|:\omega^+F \ \eta\text{-nontrivial}\}.$$

Notice that κ_0 $[\lambda_0]$ corresponds to the connectivity κ [edge-connectivity λ], and κ_1 $[\lambda_1]$ measures the superconnectivity [edge-superconnectivity] of G. When $\kappa_1>\delta$ $[\lambda_1>\delta]$ means that all the minimum disconnecting sets of order δ must be 1-trivial and, hence the digraph G is super-κ. Analogously, G is super-λ if all its minimum edge-disconnecting sets are trivial. See Boesch and Tindell [5].

Similar notations and concepts apply for (undirected) graphs. Now, all the introduced concepts are unsigned. For instance, the *boundary* of a subset of vertices F is $\partial F=\Gamma(F)\backslash F$. The *edge-boundary*, $\omega F=\{\{x,y\}\in E\colon x\in F \text{ and } y\in V\backslash F\}$. Given an integer $\eta\geqslant 0$, let us say that ∂F is *η-trivial* if there exists a subset $F'\subset F$, $F'\neq\emptyset$, with cardinality at most η, such that ∂F contains the set $\partial F'$. Analogously, the set of edges ωF is said to be *η-trivial* if ωF contains a set $\omega F'$, for some $F' \subset F$ with $|F'| \leqslant \eta$.

In what follows, it is supposed that, for the (di)graphs considered, such a κ_η or λ_η exists. Otherwise, it can be assumed, by convention, some kind of optimality for such a value (as the case of the complete digraph is dealt with respect to the standard connectivity κ_0.) Moreover, note that if ∂^+F $[\partial^-F]$ is η-nontrivial for a given η, then ∂^+F $[\partial^-F]$ is also η'-nontrivial for any $\eta' \leqslant \eta$. Thus, $\kappa(\eta') \leqslant \kappa(\eta)$ $[\lambda(\eta') \leqslant \lambda(\eta).]$

A simple connected digraph G with diameter D is said to be *s-geodetic*, $1 \leqslant s \leqslant D$, if for any $x, y \in V(G)$ there exists at most one $x \to y$ path of length less than or equal to s. When $s = D$, the digraph G is called *strongly geodetic*, see Bosák et al. [4], and Plesnik and Znám [13]. Notice that $1 \leqslant s \leqslant g-1$, since G has no loops, where g stands for the girth of the digraph. In the undirected case we have that $s = \lfloor (g-1)/2 \rfloor$.

Soneoka, Nakada, Imase and Peyrat [14,15] and Fàbrega and Fiol [7] have given sufficient conditions, in terms of this parameter s and the diameter, for a (di)graph to be maximally connected. For an s-geodetic (di)graph these sufficient conditions can be stated in the following way. If G has minimum degree δ, diameter D, connectivity $\kappa_0 = \kappa$, and edge-connectivity $\lambda_0 = \lambda$, then

$$\begin{aligned} \kappa_0 &= \delta \quad \text{if } D \leqslant 2s-1, \\ \lambda_0 &= \delta \quad \text{if } D \leqslant 2s. \end{aligned} \tag{1}$$

Let G be a maximally connected (di)graph with minimum degree $\delta > 2$, that is, $\kappa = \delta$. If $G \neq K_{\delta+1}$ it is proved in [1,9] that if G is s-geodetic with diameter D, then

$$\begin{aligned} \kappa_1 &\geqslant 2\delta - 2 \quad \text{if } D \leqslant 2s-2, \\ \lambda_1 &\geqslant 2\delta - 2 \quad \text{if } D \leqslant 2s-1. \end{aligned} \tag{2}$$

If we have no further information about the structure of G, this result is best possible in the following sense. Suppose that G contains a digon with endvertices u and v of degree δ and such that $\Gamma^+(u) \cap \Gamma^+(v) = \emptyset$. Then, the set $T = \Gamma^+(u) \cup \Gamma^+(v) \backslash \{u, v\}$ could be an example of 1-nontrivial disconnecting set with $2\delta - 2$ vertices. Thus, for such a graph G, $\kappa_1 \leqslant 2\delta - 2$ and, by the results given in (2), $D \leqslant 2s-2$ is a sufficient condition for $\kappa_1 = 2\delta - 2$. The edge case can be discussed similarly.

Given a graph G and a graph-theoretic property $\mathscr{P}$, Harary defined in [10] the *conditional connectivity* $\kappa(G; \mathscr{P})$ [edge-connectivity $\lambda(G; \mathscr{P})$] as the minimum cardinality of a set of vertices [edges], if any, whose deletion disconnects the graph and every remaining component has property $\mathscr{P}$. Notice that, for the undirected case, the parameters κ_η, λ_η measure the $\mathscr{P}_\eta$ connectivity, where $\mathscr{P}_\eta$ denotes the property of having more than η vertices. In [8] this conditional connectivity was considered only for graphs, and upper bounds on the diameter to assure optimal values for κ_η and λ_η were given. In [2] these upper bounds were improved. More precisely, the following sufficient conditions for κ_η $[\lambda_\eta]$ to be optimal were stated. Let η be an integer and let G be a graph with girth $g \geqslant \eta + 5$, minimum degree $\delta \geqslant 3$ and diameter D. Then, with $s = \lfloor (g-1)/2 \rfloor$, we have that

$$
\begin{aligned}
&\kappa_\eta \geqslant (\eta+1)\delta - 2\eta \quad \text{if } D \leqslant \begin{cases} 2s-5 & (3\leqslant \eta \leqslant \delta+2),\\ 2s-7 & (\delta+3\leqslant \eta\leqslant 2\delta+1),\\ 2s-9 & (2\delta+2\leqslant\eta\leqslant 2\delta+3),\\ 2s-5-(\eta-2\delta) & (\eta\geqslant 2\delta+4),\\ 2s-4-(\eta-2\delta) & (\eta\geqslant 2\delta+5, \eta \text{ odd}),\end{cases}\\
&\lambda_\eta \geqslant (\eta+1)\delta - 2\eta \quad \text{if } D \leqslant \begin{cases} 2s-4 & (3\leqslant \eta \leqslant \delta+2),\\ 2s-6 & (\delta+3\leqslant \eta\leqslant 2\delta+1),\\ 2s-8 & (2\delta+2\leqslant\eta\leqslant 2\delta+3),\\ 2s-4-(\eta-2\delta) & (\eta\geqslant 2\delta+4),\\ 2s-3-(\eta-2\delta) & (\eta\geqslant 2\delta+5, \eta \text{ odd}).\end{cases}
\end{aligned}
\tag{3}
$$

In the following section we introduce the parameters κ_η and λ_η, $\eta\geqslant 2$, of a digraph G from the point of view of conditional connectivity and, we find optimal lower bounds for κ_η and λ_η in the directed case. Afterwards, we state similar sufficient conditions to those of (3) for s-geodetic digraphs in order to get κ_η and λ_η to be optimal for $\eta\geqslant 2$. These conditions will now relate the parameter s, the minimum degree δ, and the diameter D. Finally, the conditions given for graphs in (3) are improved in the last section for values of $\eta\geqslant 2\delta+2$.

2. η-Fragments of digraphs and graphs

We will say that a subset F of vertices of a strongly connected digraph G is a *positive η-fragment* of G if ∂^+F is η-nontrivial, $|\partial^+F|=\kappa_\eta$ and $\overline{F}\neq\emptyset$, where $\overline{F}=V\backslash(F\cup\partial^+F)$. Analogously it is defined as a *negative η-fragment*. Note that F is a positive η-fragment if and only if $\overline{F}$ is a negative η-fragment, and $\partial^+F=\partial^-\overline{F}$. The set of vertices F is called a *positive α_η-fragment* of G if ω^+F is η-nontrivial and $|\omega^+F|=\lambda_\eta$. A *negative α_η-fragment* is defined similarly. Note that F is a positive α_η-fragment if and only if $V\backslash F$ is a negative α_η-fragment, and $\omega^+F=\omega^-(V\backslash F)$. A positive η-fragment with minimum cardinality is called *a positive η-atom*, and analogously is defined as *a negative η-atom*. All this terminology is inspired in the concepts of fragment and α-fragment introduced by Hamidoune in [11,12], in the context of standard connectivity.

Let us denote by $\delta_A^+(x)$, $\delta_A^-(x)$ the minimum out-degree and in-degree, respectively, of a vertex x in a subdigraph A of G. We have the following result.

Proposition 2.1. *Let G be a digraph, $\eta\geqslant 1$ an integer and F a positive [negative] η-fragment. Then $|F|\geqslant\eta+1$, and there exists a subset $C\subset F$, with $\partial^+C=\partial^+F$, [$\partial^-C=\partial^-F$] which is a strongly connected positive [negative] η-fragment. These results also hold for positive [negative] α_η-fragments.*

Proof. Suppose that F is a positive η-fragment. It is obvious that $|F|\geqslant\eta+1$, since otherwise ∂^+F would be η-trivial. For the same reason we have that for any vertex $x\in F$, $\delta_F^+(x)\geqslant 1$. Suppose that there exists a vertex $x\in F$ such that $\delta_F^-(x)=0$. Consider $F|\{x\}=F_1$. Then $\partial^+F_1\subset\partial^+F$, and hence, $|\partial^+F_1|\leqslant|\partial^+F|\leqslant\kappa_\eta$. Moreover, $|F_1|\geqslant\eta+1$, since ∂^+F is a η-nontrivial set. On the other hand, ∂^+F_1 must be a η-nontrivial set, and $F_1\cup\partial^+F_1\neq V(G)$. From this, we deduce that $|\partial^+F_1|\geqslant\kappa_\eta$. Then $\partial^+F_1=\partial^+F$ and F_1 is a positive η-fragment. If for each vertex $x\in F_1$, $\delta_{F_1}^-(x)\geqslant 1$, then, either F_1 is strongly connected, or is the union of strongly connected components. Otherwise, if there exists a vertex $x\in F_1$, with $\delta_{F_1}^-(x)=0$, we repeat the above reasoning. In any case, we must obtain in a finite number of steps a positive η-fragment $F'\subset F$, with $\partial^+F'=\partial^+F$, such that $\delta_{F'}^-(x)\geqslant 1$ for each vertex $x\in F'$, that is, either F' is strongly connected, or is the union of strongly connected components. Finally, assume that F' is the union of strongly connected components. In this case, there exists some component $C\subset F'$ such that $\partial^+C\subset\partial^+F'$. Then, $|\partial^+C|\leqslant|\partial^+F|\leqslant\kappa_\eta$, and hence, $|C|\geqslant\eta+1$, and ∂^+C must be a η-nontrivial set. From this, we deduce that $|\partial^+C|\geqslant\kappa_\eta$, which implies that $\partial^+C=\partial^+F$ and C is also a positive η-fragment.

The reasoning for positive [negative] α_η-fragments is similar. □

As an immediate consequence of the above proposition, a positive [negative] η-atom of a digraph has minimum cardinality $\eta+1$ and have to be strongly connected. Another consequence is that we can see κ_η [λ_η] as the minimum cardinality of a η-nontrivial set of vertices [edges], if any, whose deletion disconnects the digraph in such a way that at least two remaining strong components have more than η vertices. Furthermore, we have the following corollary.

Corollary 2.1. *Let $\eta\geqslant 1$ be integer and let G be a digraph with girth $g\geqslant\eta+1$. Then all strongly connected components contained in a positive [negative] η-fragment have more that η vertices. These results also hold for positive [negative] α_η-fragments.*

Therefore, if $g\geqslant\eta+1$, κ_η [λ_η] is the minimum cardinality of a η-nontrivial set of vertices [edges], whose deletion disconnects the digraph in such a way that all remaining strong components have more than η vertices. Suppose that a cycle $C_{\eta+1}$ with $\eta+1$ vertices ($\eta\geqslant 1$), each of degree δ in G is a subdigraph of G. Then, it is clear that $\kappa_\eta\leqslant|\partial^+C_{\eta+1}|\leqslant(\eta+1)(\delta-1)[\lambda_n\leqslant|\omega^+C_{\eta+1}|\leqslant(\eta+1)(\delta-1)]$. Hence, the value $\tau(\eta)=(\eta+1)(\delta-1)$ gives the maximum number of vertices [edges] of the positive boundary of the cycle $C_{\eta+1}$. So, $\tau(\eta)$ is the optimal value of the η-extraconnectivy. In particular, $\tau(1)=2\delta-2$ is the optimum superconnectivy of a digraph.

For graphs, all concepts related to η-fragments are similar to those of directed case, but they are unsigned. For instance, we will say that a subset F of vertices of a graph G is an *η-fragment* if ∂F is η-nontrivial, $|\partial F|=\kappa_\eta$, and $\overline{F}\neq\emptyset$, where $\overline{F}=V\backslash(F\cup\partial F)$. Obviously, $\overline{F}$ is an η-fragment as well, and $\partial\overline{F}=\partial F$. Analogously, any two α_η-fragments satisfy that $\omega F=\omega(V(G)\backslash F)$. The following proposition inform about structure of η-fragments and its proof is straightforward.

Proposition 2.2. *Let G be a graph, and F an η-fragment. Then, either F is connected or is the union of some connected components C, which satisfy that $|C| \geqslant \eta + 1$ and $\partial C = \partial F$. Furthermore, if F is an α_η-fragment then F and $V(G)\backslash F$ are connected, and $|F| \geqslant \eta + 1$, $|V(G)\backslash F| \geqslant \eta + 1$.*

So, we will refer to the components of a graph as *η-components* or *α_η-components.* Therefore, we can supppose that either any η-component or α_η-component contains a cycle, or it is a tree. In [2] we showed that $\tau(\eta) = (\eta + 1)\delta - 2\eta$ gives the maximum number of vertices of the neighborhood of a tree T with $\eta + 1$ vertices, each of degree δ, and so, it is the optimal value of the η-extraconnectivity of a graph G.

Let us define a (di)graph G as *optimally η-extraconnected* if the order of every η-nontrivial set of vertices is at least $\tau(\eta)$, that is, $\kappa_\eta \geqslant \tau(\eta)$. Analogously, we will say that a (di)graph G is *optimally edge η-extraconnected* if $\lambda_\eta \geqslant \tau(\eta)$. In the next sections we find sufficient conditions to assure optimal η-extraconnected (di)graphs. With this aim, given a digraph G, let us define the *deepness* of a positive η-fragment F, as $\mu(F) = \max_{x \in F} d(x, \partial^+ F)$. Similarly, the deepness of a negative η-fragment F is $\mu(F) = \max_{x \in F} d(\partial^- F, x)$. With respect to α_η-fragments, the *deepness* of a positive α_η-fragment F, is $\nu(F) = \max_{x \in F} d(x, \omega^+ F)$. Similarly, the deepness of a negative α_η-fragment F is $\nu(F) = \max_{x \in F} d(\omega^- F, x)$. In the same way, we define the deepness of η-components or α_η-components of a graph.

3. η-Extraconnectivity of s-geodetic digraphs

Our purpose is to state sufficient conditions on the diameter to assure optimal extraconnectivity in s-geodetic digraphs.

Let T be a tree contained in the digraph G. Then, T is said to be an *out-rooted tree with root z* if for each $v \in V(T)$ there is a unique $z \to v$ path in T. Analogously, T is said to be an *in-rooted tree with root z* if for each $v \in V(T)$ there is a unique $v \to z$ path in T. If T is an out-rooted [in-rooted] tree with root z and v is a vertex of T, then *the level number of v* is the length of the unique $z \to v$ [$v \to z$] path in T. The root z is at level 0. The number of levels of T is called *the height* of T and it is denoted by h_T.

Let T a subdigraph of G. For each vertex $t \in V(T)$ let us denote by $N_T^+(t) = \Gamma^+(t)\backslash V(T)$. We have the following lemma.

Lemma 3.1. *Let G be an s-geodetic digraph with minimum degree δ, which contains an out-rooted [in-rooted] tree T with root in a vertex z, and height h_T.*

(a) *If $h_T \leqslant s - 1$ then $|\partial^+ T| \geqslant (\delta - 1)|T|$ [$|\partial^- T| \geqslant (\delta - 1)|T|$].*

(b) *If T is contained in a positive [negative] η-fragment F with deepness $\mu(F) = \mu$, and $h_T + \mu + 1 \leqslant s$ then $|\partial^+ T| \leqslant |\partial^+ F|$ [$|\partial^- T| \leqslant |\partial^- F|$].*

(c) *If T is contained in a positive [negative] α_η-fragment F with deepness $\nu(F) = \nu$, and $h_T + \nu + 2 \leqslant s$ then $|\partial^+ T| \leqslant |\omega^+ F|$ [$|\partial^- T| \leqslant |\omega^- F|$].*

Proof. (a) Suppose that T is an out-rooted tree with root in a vertex z, and assume that there exists a vertex $u \in \Gamma^+(t_1) \cap \Gamma^+(t_2)$, for two distinct vertices $t_1, t_2 \in V(T)$. Then, there are two different paths from the root z to vertex u, namely, $z \to t_1 \to u$ and $z \to t_2 \to u$, whose lengths are at most $h_T + 1 \leqslant s$. This is a contradiction, since the digraph is s-geodetic. Therefore, $N_T^+(t_1) \cap N_T^+(t_2) = \emptyset$, for two distinct vertices $t_1, t_2 \in V(T)$, and hence, $|\partial^+ T| = \sum_{t \in V(T)} |N_T^+(t)| \geqslant \sum_{t \in V(T)} (\delta - \delta_T^+(t)) \geqslant (\delta - 1)|T|$, since $\sum_{t \in V(T)} \delta_T^+(t) \leqslant |T|$.

The reasoning is similar for in-rooted trees.

(b) Suppose that T is an out-rooted tree with root in a vertex z contained in a positive η-fragment F, and assume that $|\partial^+ T| > |\partial^+ F|$. Then, there exist two distinct vertices $u, v \in \partial^+ T$, at minimum distance from the same vertex $q \in \partial^+ F$, that is, $d(u, \partial^+ F) = d(u, q) \leqslant \mu$, and $d(v, \partial^+ F) = d(v, q) \leqslant \mu$. (Notice that it could be $u \in \partial^+ F$ or $v \in \partial^+ F$. In this case $u = q$ or $v = q$.) On the other hand, the root z of T satisfies $d(z, u) \leqslant h_T + 1 \leqslant s - \mu$, and also $d(z, v) \leqslant s - \mu$. Therefore, we have two differents paths from z to q of length at most s, namely, $z \to u \to q$ and $z \to v \to q$, which is a contradiction.

The reasoning is similar for in-rooted trees contained in negative η-fragments.

(c) Suppose that T is an out-rooted tree with root in a vertex z contained in a positive α_η-fragment F, and assume that $|\partial^+ T| > |\omega^+ F|$. Let us denote by $U = \{u \in F, (u, v) \in \omega^+ F\}$ and $W = \{v \in V \backslash F, (u, v) \in \omega^+ F\}$. Obviously, $\max(|U|, |W|) \leqslant |\omega^+ F|$, and for each $x \in F$, $d(x, \omega^+ F) = d(x, U)$, that is, $\max_{x \in F} d(x, U) = v$. Moreover, if $u \in \partial^+ T$ then the path $z \to u$ has length at most $h_T + 1 < s$. This implies that for each $u \in \partial^+ T$, it is $u \notin \Gamma^-(\partial^+ T) \cup \Gamma^+(\partial^+ T)$. On the contrary, there would exist some vertex $y \in \partial^+ T$ such that either $u \in \Gamma^-(y)$ or $u \in \Gamma^+(y)$. Hence, we would have either the paths $z \to y$, of length at most $h_T + 1 < s$, and $z \to uy$, of length at most $h_T + 2 \leqslant s$, or the paths $z \to yu$ of length at most $h_T + 2 \leqslant s$, and $z \to u$ of length at most $h_T + 1 < s$. In any case, we would have a contradiction. From this, we deduce that $|\partial^+ T \cap U| + |\Gamma^-(\partial^+ T \cap W) \cap U| \leqslant |U|$ and also $|\partial^+ T \cap W| + |\Gamma^+(\partial^+ T \cap U) \cap W| \leqslant |W|$. Since $|\partial^+ T \cap U| \leqslant |\Gamma^+(\partial^+ T \cap U) \cap W|$ and $|\partial^+ T \cap W| \leqslant |\Gamma^-(\partial^+ T \cap W) \cap U|$, we can conclude that $|\partial^+ T \cap (U \cup W)| \leqslant \max(|U|, |W|)$. Therefore, as $|\partial^+ T| > \max(|U|, |W|)$, there exist two distinct vertices $u, v \in \partial^+ T$ with either u or v belonging to $F \backslash U$, which are at minimum distance from the same vertex $q \in U \cup W$. Again, we have two different paths of lengths at most $(h_T + 1) + v + 1 \leqslant s$, a contradiction.

The reasoning is similar for in-rooted trees contained in negative α_η-fragments. $\square$

From now on, we will suppose that $s \geqslant \eta + 1$. Since, $s \leqslant g - 1$, where g denotes the girth of digraph, it is $g > \eta + 1$, and hence, all strongly connected components contained in a positive [negative] η-fragments or α_η-fragment have more than η vertices. In the following lemma, we compute the minimum deepness of η-fragments and α_η-fragments.

Lemma 3.2. *Let $\eta \geqslant 2$ be an integer and G an s-geodetic digraph with $s \geqslant \eta + 1$, and minimum degree $\delta \geqslant 3$. Then,*

(a) $\mu(F)\geqslant 2$ *if* F *is a positive or negative* η*-fragment and* $\kappa_\eta<\tau(\eta)$;
(b) $\nu(F)\geqslant 1$ *if* F *is a positive or negative* α_η*-fragment and* $\lambda_\eta<\tau(\eta)$.

Proof. (a) Consider a positive η-fragment F and suppose that $\mu=\mu(F)=1$. By Corollary 2.1 we can consider a path $P=x_0x_1\ldots x_\eta$, of length η contained in F, because $\eta+1\leqslant s\leqslant g-1$, where g denotes the girth of the digraph. First of all, notice that $N_P^+(x_i)\cap N_P^+(x_j)=\emptyset$, $0\leqslant i<j\leqslant\eta$, since $s\geqslant\eta+1$, that is, the paths of length at most $\eta+1$ are uniques. For each vertex $v\in N_P^+(x_i)$ let $q_v\in\partial^+F$ be a vertex at minimum distance from v, and let $Q_i\subset\partial^+F$ be the set of such vertices q_v. Notice that either $v=q_v$ or $d(v,q_v)=1$. Since $s\geqslant\eta+1$, we have that $|Q_i|\geqslant\delta-1$, for $0\leqslant i\leqslant\eta-1$ and $|Q_\eta|\geqslant\delta$. Moreover, $Q_i\cap Q_j=\emptyset$ for $1\leqslant j-i\leqslant\eta-1$, $0\leqslant i<j\leqslant\eta$. As $|\partial^+F|\leqslant(\eta+1)(\delta-1)-1$ we deduce that $|Q_0\cap Q_\eta|\geqslant 2$. This implies that there exist at least two vertices $q,q'\in Q_0\subset\partial^+F$ at minimum distance from two vertices $y,z\in N_P^+(x_\eta)$. That is, $d(x_0,q)\leqslant 2$, $d(x_0,q')\leqslant 2$, and so, the length of the paths Pyq, Pzq', which are at most $\eta+2$, must be greater than s. Since, $s\geqslant\eta+1\geqslant 3$, we have that the length of these paths must be $\eta+2=s+1$, or equivalently, $\eta+1=s$, which means that the vertices $y,z\in F$.

The paths $P_y=x_1\ldots x_\eta y$, and $P_z=x_1\ldots x_\eta z$, are contained in F, and reasoning in the same way as for path P, we find two vertices belonging to $N_{P_y}^+(y)\cap F$ and two vertices belonging to $N_{P_z}^+(z)\cap F$. Continuing in this way, we can construct in F an out-rooted tree T with root x_η, degree at least 2 and height $h_T=s-2=\eta-1$, because the digraph is s-geodetic and $s\leqslant g-1$, This tree T obviously satisfies that $h_T+\mu+1\leqslant s$, and $|T|=\sum_{i=0}^{s-2}2^i=2^{s-1}-1$. Therefore, from Lemma 3.1 we deduce that $(2^{s-1}-1)(\delta-1)\leqslant|\partial^+T|\leqslant|\partial^+F|\leqslant(\eta+1)(\delta-1)-1=s(\delta-1)-1$, which is a contradiction, because $(2^{s-1}-1)(\delta-1)>s(\delta-1)-1$, for $s\geqslant 3$.

(b) Consider now a positive α_η-fragment, F, with $\nu(F)=0$ and $\lambda_\eta<\tau(\eta)$. As in the above case, we can consider that a path $P=x_0x_1\ldots x_\eta$ of length η is contained in F. Then, for each $x\in F$ let us denote by $E^+(x)=\{(x,y)\in\omega^+F\}$. As $\nu(F)=0$, we have that $|E^+(x)|\geqslant 1$ for each $x\in F$. Therefore, $|E^+(x_i)|+|N_P^+(x_i)\cap F|\geqslant\delta-1$, for $0\leqslant i\leqslant\eta$. As $s\geqslant\eta+1$, it is satisfied that $\Gamma^+(x_i)\cap\Gamma^+(x_j)=\emptyset$, if $i\neq j$. Hence, $|\omega^+F|\geqslant\sum_{i=0}^{\eta}[|E^+(x_i)|+\sum_{z\in N_P^+(x_i)\cap F}|E^+(z)|]\geqslant(\eta+1)(\delta-1)$, which is a contradiction with the hypothesis. □

In the following lemma, we state that the deepness of η-fragments and α_η-fragments increases if the values of η are large enough with respect to minimum degree δ.

Lemma 3.3. *Let* G *be an s-geodetic digraph with* $s\geqslant\eta+1$, *minimum degree* $\delta\geqslant 3$, *and* $\eta\geqslant 1+\delta$. *Then,*
(a) $\mu(F)\geqslant\lfloor\log_\delta(\eta(\delta-1)+1)\rfloor+1$, *if* F *is a positive or negative* η*-fragment and* $\kappa_\eta<\tau(\eta)$;
(b) $\nu(F)\geqslant\lfloor\log_\delta(\eta(\delta-1)+1)\rfloor$, *if* F *is a positive or negative* α_η*-fragment and* $\lambda_\eta<\tau(\eta)$.

Proof. (a) Let F be a positive η-fragment. Under hypothesis assumed, we can apply Lemma 3.2 and then $\mu(F)\geqslant 2$. As $\eta\geqslant 1+\delta$ we can suppose that there exists an integer

$k \geqslant 1$ such that $1 + \delta + \delta^2 + \cdots + \delta^k \leqslant \eta < 1 + \delta + \delta^2 + \cdots + \delta^{k+1}$. From this, we can deduce that $k + 1 = \lfloor \log_\delta(\eta(\delta - 1) + 1) \rfloor$. On the other hand, we have that $\log_\delta x \leqslant x/\delta$, for each $x \geqslant \delta$. Then

$$k + 1 \leqslant \left\lfloor \frac{\eta(\delta - 1) + 1}{\delta} \right\rfloor \leqslant s - 1 - \left\lfloor \frac{s - 2}{\delta} \right\rfloor \leqslant s - 1,$$

since $s \geqslant \eta + 1 \geqslant \eta \geqslant \delta + 2$. We are going to reason by contradiction, and so, assume that $2 \leqslant \mu \leqslant k + 1$. Let $z \in F$ a vertex such that $d(z, \partial^+ F) = \mu$ and denote by S_z the tree formed by z and δ of its out-neighbours. For each vertex $x \in S_z$, $\delta_F^+(x) \geqslant 1$ because $\partial^+ F$ is a η-nontrivial set. Therefore, we can suppose that F contains δ paths starting in z of length $h \geqslant 2$, where $(h - 1)\delta + 2 \leqslant \eta + 1 \leqslant h\delta + 1$, because

$$h = \left\lceil \frac{\eta}{\delta} \right\rceil \leqslant \left\lfloor \frac{\eta - 1}{\delta} \right\rfloor + 1 \leqslant \left\lfloor \frac{s - 2}{\delta} \right\rfloor + 1 \leqslant s - 1,$$

since $\delta \geqslant 3$, and $s \leqslant g - 1$, where g denotes the girth of the digraph. This structure is an out-rooted tree T with root in vertex z, whose number of levels is $h_T = h \leqslant s - 1$, and $|T| = h\delta + 1 \geqslant \eta + 1$. Therefore, by applying Lemma 3.1 we obtain that $|\partial^+ T| \geqslant (\delta - 1)|T| \geqslant (\delta - 1)(\eta + 1)$. Furthermore, we have that

$$\begin{aligned} h_T + \mu + 1 \leqslant h_T + (k + 1) + 1 &\leqslant \left\lfloor \frac{\eta - 1}{\delta} \right\rfloor + 1 + \lfloor \log_\delta(\eta(\delta - 1) + 1) \rfloor + 1 \\ &\leqslant \left\lfloor \frac{\eta - 1 + \delta}{\delta} \right\rfloor + \left\lfloor \frac{\eta(\delta - 1) + 1 - \delta}{\delta} \right\rfloor + 1 \\ &\leqslant \eta + 1 \leqslant s, \end{aligned}$$

since $\log_\delta x \leqslant (x - \delta)/\delta$, for any $x \geqslant \delta^2$, because $\delta \geqslant 3$, and $\eta(\delta - 1) + 1 \geqslant (\delta + 1)(\delta - 1) + 1 = \delta^2$. By applying Lemma 3.1 we obtain that $(\delta - 1)(\eta + 1) \leqslant |\partial^+ T| \leqslant |\partial^+ F| \leqslant \kappa_\eta < \tau(\eta)$, which is a contradiction.

For negatives η-fragments the argument is similar.

(b) Now, we apply Lemma 3.2 to a positive α_η-fragment F, and then $v = v(F) \geqslant 1$. As in case (a) we can suppose that there exists an integer $k \geqslant 1$ such that $1 + \delta + \delta^2 + \cdots + \delta^k \leqslant \eta \leqslant \delta + \delta^2 + \cdots + \delta^{k+1}$. Therefore, as before we have that $k + 1 = \lfloor \log_\delta(\eta(\delta - 1) + 1) \rfloor \leqslant s - 1$. We are going to reason by contradiction, and so, assume that $1 \leqslant v \leqslant k$. Consider a vertex $z \in F$ such that $d(z, \omega^+ F) = v$ and, in the same way as before, we can suppose that F contains the same out-rooted tree T with root in vertex z as in case (a), with height $h_T = h \leqslant s - 1$, and $|T| \geqslant \eta + 1$. Consequently, we have again that $h_T + v + 2 \leqslant h_T + k + 2 \leqslant \eta + 1 \leqslant s$. By applying Lemma 3.1 we obtain that $(\delta - 1)(\eta + 1) \leqslant |\partial^+ T| \leqslant |\omega^+ F| \leqslant \lambda_\eta < \tau(\eta)$, which is a contradiction.

For negatives α_η-fragments the argument is similar. □

Next, we state our main theorem, in which we give upper bounds on the diameter to assure that the extraconnectivity of a s-geodetic digraph is optimum.

Theorem 3.1. *Let $\eta \geqslant 2$ be an integer and G a s-geodetic digraph with $s \geqslant \eta + 1$, minimum degree $\delta \geqslant 3$ and diameter D. Then,*

(a) $\kappa_\eta \geqslant \tau(\eta)$ *if* $D \leqslant 2s - 1 - 2\lfloor \log_\delta(\eta(\delta - 1) + 1) \rfloor$;

(b) $\lambda_\eta \geqslant \tau(\eta)$ *if* $D \leqslant 2s - 2\lfloor \log_\delta(\eta(\delta - 1) + 1) \rfloor$.

Proof. (a) Assume that $\kappa_\eta < \tau(\eta)$, and let F be an η-positive fragment, $\overline{F} = V(G) \backslash (F \cup \partial^+ F)$. First, assume that $2 \leqslant \eta \leqslant \delta$. Then $\lfloor \log_\delta(\eta(\delta - 1) + 1) \rfloor = 1$, and hence, $D \leqslant 2s - 3$. Consider $z \in F$ and $y \in \overline{F}$ at maximum distance to and from $\partial^+ F$. It is clear that $D \geqslant d(z, y) \geqslant d(z, \partial^+ F) + d(\partial^+ F, y) = \mu(F) + \mu(\overline{F})$. If $\mu = \mu(F) \leqslant \mu(\overline{F})$, we must have $\mu \leqslant s - 2$. Moreover, from Lemma 3.2 it is $\mu \geqslant 2$, and hence, the η-positive fragment F contains the tree S_z formed by z and δ of its out-neighbours. The height of this tree is obviously $h_{S_z} = 1$, and it satisfies $h_{S_z} + \mu + 1 \leqslant s$. By applying Lemma 3.1 we obtain that $(\delta - 1)(\delta + 1) \leqslant |\partial^+ T| \leqslant |\partial^+ F| = \kappa_\eta < \tau(\eta) = (\eta + 1)(\delta - 1)$, which is a contradiction since $2 \leqslant \eta \leqslant \delta$.

Secondly, suppose $\eta \geqslant \delta + 1$. Reasoning as in the above case we have that $\mu = \mu(F) \leqslant s - 1 - \lfloor \log_\delta(\eta(\delta - 1) + 1) \rfloor$. Let $k \geqslant 1$ be an integer such that $1 + \delta + \delta^2 + \cdots + \delta^k \leqslant \eta \leqslant \delta + \delta^2 + \cdots + \delta^{k+1}$, that is, $k + 1 = \lfloor \log_\delta(\eta(\delta - 1) + 1) \rfloor$. From Lemma 3.3 we know that $\mu \geqslant k + 2$. Therefore, F contains an out-rooted tree T with root in vertex z such that $d(z, \partial^+ F) = \mu$, with height $h_T = k + 1 < \mu < s - 1$, and hence, its internal vertices have minimum degree δ. Then, $|T| \geqslant \sum_{i=0}^{k+1} \delta^i \geqslant \eta + 1$, and from Lemma 3.1 we deduce that $|\partial^+ T| \geqslant (\delta - 1)(\eta + 1)$. Moreover, we also have that $h_T + \mu + 1 \leqslant k + 1 + s - 1 - \lfloor \log_\delta(\eta(\delta - 1) + 1) \rfloor + 1 \leqslant s$, since $k + 1 \leqslant \lfloor \log_\delta(\eta(\delta - 1) + 1) \rfloor$. Therefore, from Lemma 3.1 we deduce again that $(\delta - 1)(\eta + 1) \leqslant |\partial^+ T| \leqslant |\partial^+ F| = \kappa_\eta < \tau(\eta)$, which is a contradiction.

(b) Now, let F be an α_η-positive fragment and assume that $\lambda_n < \tau(\eta)$. First, for $2 \leqslant \eta \leqslant \delta$ it is $D \leqslant 2s - 2$. Let us consider two vertices, $z \in F$ and $y \in V(G) \backslash F$ at maximum distance to and from $\omega^+ F$. It is clear that $D \geqslant d(z, y) \geqslant d(z, \omega^+ F) + 1 + d(\omega^+ F, y) = v(F) + 1 + v(V(G) \backslash F)$. If $v = v(F) \leqslant v(V(G) \backslash F)$, we must have $v \leqslant s - 2$. Moreover, from Lemma 3.2 it is $v \geqslant 1$, and hence, the α_η-positive fragment F contains the tree S_z formed by z and δ of its out-neighbours. The number of levels of this tree is obviously $h_{S_z} = 1$, and it is satisfied that $h_{S_z} + v + 1 \leqslant s$. By applying Lemma 3.1 we obtain that $(\delta - 1)(\delta + 1) \leqslant |\partial^+ T| \leqslant |\omega^+ F| = \lambda_\eta < \tau(\eta)$, which is a contradiction since $2 \leqslant \eta \leqslant \delta$.

Secondly, for $\eta \geqslant \delta + 1$, reasoning as before, we have that $v = v(F) \leqslant s - 2 - \lfloor \log_\delta(\eta(\delta - 1) + 1) \rfloor$. Let $k \geqslant 1$ an integer such that $1 + \delta + \delta^2 + \cdots + \delta^k \leqslant \eta \leqslant \delta + \delta^2 + \cdots + \delta^{k+1}$, that is, $k + 1 = \lfloor \log_\delta(\eta(\delta - 1) + 1) \rfloor$. From Lemma 3.3 we can deduce that $v \geqslant k + 1$. From now on, we can reason as in vertex case, and so, F contains an out-rooted tree T with root in a vertex z such that $d(z, \omega^+ F) = v$, with height $h_T = k + 1 < v < s - 1$, and hence, its internal vertices have minimum degree δ. Then, $|\partial^+ T| \geqslant (\delta - 1)|T| \geqslant (\delta - 1)(\eta + 1)$. Moreover, we have also that $h_T + v + 2 \leqslant k + 1 + s - 2 - \lfloor \log_\delta(\eta(\delta - 1) + 2) \rfloor + 1 \leqslant s$. Therefore, from Lemma 3.1 we deduce again that $(\delta - 1)(\eta + 1) \leqslant |\partial^+ T| \leqslant |\omega^+ F| = \lambda_\eta < \tau(\eta)$, which is a contradiction. □

4. η-Extraconnectivity of graphs with large girth

In this section we will only consider connected graphs, and the objective is to improve the conditions given in (3).

Let T be a tree contained in the graph G, and let us denote its diameter by D_T. As in the above section, a tree T is said to be a *rooted tree with root* z if for each $v \in V(T)$ there is a unique $z \leftrightarrow v$ path in T. If T is a rooted tree with root z and v is a vertex of T, then the level number of v is the length of the unique $z \leftrightarrow v$ path in T, which is denoted by $d_T(z,v)$. The root z is at level 0, and $h_T = \max_{v \in V(T)} d_T(z,v)$ denotes the number of levels. Clearly, the diameter of a rooted tree is $D_T \leqslant 2h_T$.

Furthermore, the following concepts and notations were introduced in [2]. Let T be a tree contained in a given η-component C. For every vertex v of T we will consider a path $T^*(v) = v_0 v_1 \dots v_{s_v - 1} v_{s_v}$, $v_0 = v$, $s_v \geqslant 1$, $v_1 \notin V(T)$, such that $d(v_i, F) > d(v_{i-1}, F)$, $1 \leqslant i \leqslant s_v$, and $d(h,F) \leqslant d(v_{s_v}, F)$ for every $h \notin V(T^*(v))$ adjacent to v_{s_v} (if such a path does not exist, let $s_v = 0$ and consider the trivial path $T^*(v) = v$.) Moreover, define $N_T^*(v) = \Gamma(v_{s_v}) \backslash \{v_{s_v - 1}\}$ (if $s_v = 0$, then $N_T^*(v) = N_T(v)$) and let $N^*(T) = \bigcup_{v \in V(T)} N_T^*(v)$. For any $v \in V(T)$, let $T \oplus T^*(v)$ denote a subgraph obtained by attaching to T the path $T^*(v)$. Moreover, let T^* be the subgraph obtained by joining $T^*(v)$ to each $v \in V(T)$. That is, if $V(T) = \{v_0, v_1, \dots, v_r\}$, then $T^* = T \oplus T^*(v_0) \oplus \cdots \oplus T^*(v_r)$. If the diameter of T^*, D_{T^*}, is less than g then T^* is a tree. Notice that, in a certain sense, T^* is as far as possible from ∂C.

As in directed case, $\mu(C)$ denotes the deepness of a η-component, and $\nu(C)$ denotes the deepness of a α_η-component. We will use the following results contained in [2].

Lemma 4.1. *Let $\eta \geqslant 1$ be an integer and let G be a graph with girth $g \geqslant \eta + 5$ minimum degree $\delta \geqslant 3$. Let C any component or α_η-component.*

(a) *If C contains a tree T of order $\eta + 1$ such that $D_{T^*} < g - 2$, then $|N^*(T)| \geqslant \tau(\eta)$. Moreover, if $D_{T^*} < g - 2\mu(C) - 2$, then $|\partial C| \geqslant |N^*(T)| \geqslant \tau(\eta)$. And, if $D_{T^*} < g - 2\nu(C) - 2$, then $|\omega C| \geqslant |N^*(T)| \geqslant \tau(\eta)$.*

(b) *If $\eta \geqslant \delta + 1$ and $\kappa_\eta < \tau(\eta)$, $\lambda_\eta < \tau(\eta)$, then C satisfies*

$$\mu(C) \geqslant 3, \text{ or } \nu(C) \geqslant 3 \text{ if } \begin{cases} \delta \geqslant 5, \\ \eta - \delta + 9 < g, & \text{for } 3 \leqslant \delta \leqslant 4. \end{cases} \qquad \square$$

We proceed now to prove that the deepness of the η-components and α_η-components of a graph increases when the value of η is large enough.

Lemma 4.2. *Let G be a graph with girth $g > \eta - \delta + 9$ if minimum degree $3 \leqslant \delta \leqslant 4$, and $g \geqslant \eta + 5$ for $\delta \geqslant 5$. Suppose that the η-extraconnectivities are $\kappa_\eta < \tau(\eta)$, $\lambda_\eta < \tau(\eta)$, and that $\eta \geqslant \delta + 1$. Then any η-component or α_η-component C satisfies that*

$$\mu(C), \nu(C) \geqslant \left\lfloor \log_{(\delta - 1)} \left(\frac{(\eta - 1)(\delta - 2)}{\delta} + 1 \right) \right\rfloor + 2.$$

Proof. Let C be an η-component. As $\eta \geqslant \delta + 1$ we can suppose that there exists an integer $k \geqslant 1$ such that $1 + \delta + \delta(\delta - 1) + \cdots + \delta(\delta - 1)^{k-1} \leqslant \eta < 1 + \delta + \delta(\delta - 1) + \cdots + \delta(\delta - 1)^k$. This implies that

$$k = \left\lfloor \log_{(\delta-1)} \left(\frac{(\eta - 1)(\delta - 2)}{\delta} + 1 \right) \right\rfloor .$$

First of all, see that $2k < g - 2$. In effect,

$$2k = 2 \left\lfloor \log_{(\delta-1)} \left(\frac{(\eta - 1)(\delta - 2)}{\delta} + 1 \right) \right\rfloor \leqslant 2 \left\lfloor \frac{\eta(\delta - 2) + 2}{\delta(\delta - 1)} \right\rfloor ,$$

since $\log_{(\delta-1)} x \leqslant x/(\delta - 1)$, for each $x \geqslant \delta - 1$, and

$$\frac{(\eta - 1)(\delta - 2)}{\delta} + 1 \geqslant \delta - 1 \quad \text{for } \eta \geqslant \delta + 1.$$

Therefore, if $3 \leqslant \delta \leqslant 4$, we have by hypothesis that $\eta < g + \delta - 9$ and so, $2k \leqslant \lfloor (g - 4)/3 \rfloor < g - 2$. If $\delta \geqslant 5$ then $\eta \leqslant g - 5$, and thus, $2k \leqslant 2\lfloor (g - 5)/\delta \rfloor < g - 2$.

Now, see that $\mu(C) \geqslant k + 2$. We will prove it by induction with respect to k. When $k = 1$ we have that $1 + \delta \leqslant \eta < 1 + \delta + \delta(\delta - 1)$, and hence, the result is true by Lemma 4.1. By hypothesis of induction we may assume that the result is true for values less than k. Suppose that $\eta \geqslant 1 + \delta + \delta(\delta - 1) + \cdots + \delta(\delta - 1)^k$; by hypothesis of induction $\mu(C) \geqslant k + 1$. We are going to reason by contradiction and so assume that $\mu(C) = k + 1$. Now, take a vertex $z \in V(C)$ such that $d(z, \partial C) = \mu(C) = k + 1$. Since $2k < g$, we can consider the component of a rooted tree T' with root z, number of levels k, and all its internal vertices with degree δ. Therefore, $|T'| = 1 + \delta \sum_{i=0}^{k-1} (\delta - 1)^i < \eta + 1$, and since ∂C is a η-nontrivial set, we can consider in C a tree T of order $\eta + 1$ that contains T' and such that

$$D_T \leqslant (\eta + 1) - |T'| + D_{T'} \leqslant \eta - \delta \frac{(\delta - 1)^k - 1}{\delta - 2} + 2k.$$

Therefore, the diameter of subgraph T^* is at most $D_{T^*} \leqslant D_T + 2(\mu - 1)$, and hence,

$$D_{T^*} + 2\mu + 2 \leqslant \eta - \delta \frac{(\delta - 1)^k - 1}{\delta - 2} + 6k + 2.$$

Now, if $3 \leqslant \delta \leqslant 4$ we obtain that $D_{T^*} + 2\mu + 2 \leqslant \eta - \delta + 9 < g$, because

$$-\delta \frac{(\delta - 1)^k - 1}{\delta - 2} + 6k + 2 < -\delta + 9,$$

for any $k \geqslant 1$. If $\delta \geqslant 5$ we obtain that $D_{T^*} + 2\mu + 2 \leqslant \eta + 4 < g$, since

$$-\delta \frac{(\delta - 1)^k - 1}{\delta - 2} + 6k - 2 \leqslant 0,$$

for any $k \geqslant 1$. From this, by applying Lemma 4.1 we can deduce that $|N^*(T)| \geqslant \tau(\eta)$, since $D_{T^*} < g - 2$. Furthermore, $\kappa_\eta = |\partial C| \geqslant |N^*(T)| \geqslant \tau(\eta)$, which is a contradiction to the hypothesis. Therefore, $\mu(C) \geqslant k + 2$ as claimed.

For α_η-fragments the reasoning is the same as in vertex case. □

As stated in the Introduction, a graph G is always s-geodetic for $s = \lfloor (g-1)/2 \rfloor$, g being the girth of graph. The main result of this section is the following theorem, which is a consequence of the above lemma.

Theorem 4.1. *Let G be a graph with girth $g > \eta - \delta + 9$ if minimum degree $3 \leqslant \delta \leqslant 4$, and $g \geqslant \eta + 5$ for $\delta \geqslant 5$. If $\eta \geqslant \delta + 1$, and diameter is D, then,*
(a) $\kappa_\eta \geqslant \tau(\eta)$ *if* $D \leqslant 2s - 5 - 4\lfloor \log_{(\delta-1)}(\frac{(\eta-1)(\delta-2)}{\delta} + 1) \rfloor$;
(b) $\lambda_\eta \geqslant \tau(\eta)$ *if* $D \leqslant 2s - 4 - 4\lfloor \log_{(\delta-1)}(\frac{(\eta-1)(\delta-2)}{\delta} + 1) \rfloor$.

Proof. As $\eta \geqslant \delta + 1$, there exists an integer $k \geqslant 1$ such that $1 + \delta \sum_{i=0}^{k-1} (\delta-1)^i \leqslant \eta \leqslant \delta \sum_{i=0}^{k} (\delta-1)^i$, that is,

$$k = \left\lfloor \log_{(\delta-1)} \left(\frac{(\eta-1)(\delta-2)}{\delta} + 1 \right) \right\rfloor .$$

To prove (a) assume that $\kappa_\eta < \tau(\eta)$, and let F be a η-fragment, that is, $|\partial F| = \kappa_\eta$. By Proposition 2.2, we may consider an η-component $C \subset F$, and an η-component $\overline{C} \subset \overline{F}$ such that $\partial C = \partial \overline{C} = \partial F$. Now, take a vertex x belonging to C, and a vertex y belonging to $\overline{C}$, such that $d(x, \partial C) = \mu(C)$, and $d(\partial C, y) = \mu(\overline{C})$. It is clear that $D \geqslant d(x,y) \geqslant d(x, \partial C) + d(\partial C, y) = \mu(C) + \mu(\overline{C})$. If $\mu = \mu(C) \leqslant \mu(\overline{C})$, we must have that $\mu \leqslant s - 3 - 2k$. From Lemma 4.2 we know that $\mu \geqslant k + 2$ and $2k < g - 2$. Therefore, C contains a rooted tree T with root in vertex x, number of levels $h_T = k + 1 < \mu$, and $|T| = 1 + \delta \sum_{i=0}^{k} (\delta-1)^i \geqslant \eta + 1$. Now, we consider the subgraph T^*. Note that, by characteristics of tree T, for any given vertex v of T, the length of path $T^*(v) = v_0 v_1 \ldots v_{s_v - 1} v_{s_v}$, is at most the number of levels of T, h_T, since $d(v_i, F) > d(v_{i-1}, F)$, $0 \leqslant i \leqslant s_v$, and $d(h, F) \leqslant d(v_{s_v}, F)$ for every $h \notin V(T^*(v))$ adjacent to v_{s_v}. Hence, the diameter of subgraph T^* is at most $D_{T^*} \leqslant D_T + 2h_T = 2D_T = 4k + 4$, since $D_T = 2k + 2$. Thus, $D_{T^*} + 2\mu + 2 \leqslant (4k+4) + (2s - 6 - 4k) + 2 \leqslant 2s < g$. From Lemma 4.1 we deduce that $|N^*(T)| \geqslant \tau(\eta)$, and also that $\kappa_\eta = |\partial C| \geqslant |N^*(T)| \geqslant \tau(\eta)$, which is a contradiction. Therefore, the graph is optimally η-extraconnected, that is $\kappa_\eta \geqslant \tau(\eta)$.

(b) Assume that $\lambda_\eta < \tau(\eta)$, and let F be an α_η-fragment, that is, $|\omega F| = \lambda_\eta$. Now, we may consider a vertex x in the α_η-component F, and a vertex y in a α_η-component $V(G) \backslash F$, such that $d(x, \omega F) = v(F)$, and $d(\omega F, y) = v(V(G) \backslash F)$. It is clear that $D \geqslant d(x,y) \geqslant d(x, \omega F) + 1 + d(\omega F, y) = v(F) + 1 + v(V(G) \backslash F)$. If $v = v(F) \leqslant v(V(G) \backslash F)$, we must have that $v \leqslant s - 3 - 2k$. From now on, the reasoning is exactly the same as in the vertex-case. □

The results of Theorem 4.1 must be compared with those of (3). For instance, if $1 + \delta \leqslant \eta \leqslant \delta + \delta(\delta - 1)$, then

$$\left\lfloor \log_{(\delta-1)} \left(\frac{(\eta-1)(\delta-2)}{\delta} + 1 \right) \right\rfloor = 1,$$

and Theorem 4.1 states that $\kappa_\eta \geqslant \tau(\eta)$ if $D \leqslant 2s-9$, and also $\lambda_\eta \geqslant \tau(\eta)$ if $D \leqslant 2s-8$. Furthermore, if $1+\delta+\delta(\delta-1) \leqslant \eta \leqslant \delta+\delta(\delta-1)+\delta(\delta-1)^2$, then

$$\left\lfloor \log_{(\delta-1)}\left(\frac{(\eta-1)(\delta-2)}{\delta}+1\right)\right\rfloor = 2,$$

and Theorem 4.1 states that $\kappa_\eta \geqslant \tau(\eta)$ if $D \leqslant 2s-13$, and also $\lambda_\eta \geqslant \tau(\eta)$ if $D \leqslant 2s-12$. Hence, it is obvious that the previous known sufficient conditions given in (3) for a graph to be optimally extraconnected are improved for $\eta \geqslant 14$, if $\delta = 3$, and for $\eta \geqslant 2\delta+2$ if $\delta \geqslant 4$.

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